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# Range minimization problems in path-facility location on trees

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ABSTRACT

In this paper, we study the problem of locating path-shaped facilities on a tree network with non negative weights associated to the vertices and positive lengths associated to the edges. Our objective is to ensure low variability of the distribution of the distances from the demand points (clients) to a facility. In the location process, we take into account both the maximum and the minimum weighted distances of a client to a facility and we formulate our problem in order to minimize the "Range" function which is defined as the difference between the maximum and the minimum weighted distances from the vertices of the network to a facility. We discuss different formulations of the problem providing polynomial time algorithms for each of them. We solve in polynomial time all the above problems also when an additional constraint on the maximum length of the path is introduced.



## 1. Introduction

In location analysis, the issue of equity among clients has become a relevant criterion especially when locating public facilities. Equity refers to a fair location of the facilities w.r.t. the distribution of the clients' demand in a geographical area and the objective is to locate facilities in order to ensure a low variability of the distribution of the distances from the demand points (clients) to them. In particular, in this paper, we study the problem of locating path-shaped facilities on tree networks. Location problems on a network related to *extensive* facilities, such as paths or trees, are studied nowadays (for a comprehensive review, see, for example, [4,6,20,26]). In practice, the problem consists of locating a connected structure in a network in order to supply a set of costumers. Commonly used objective functions are the sum of the distances from each client to its nearest facility (median criterion) [8,11,16,24], or the maximum of these distances (*center* or *eccentricity* criterion) [7,9,22,23], or combinations of the two [1,2,17,27]. Also the more general ordered median objective can be adopted [10,21]. Here we consider the maximum and the minimum weighted distances of a client to a facility and minimize the *weighted Range* function which is defined as the difference between the maximum and the minimum weighted distances from the vertices of the network to the path. The problem of finding a path minimizing the weighted Range function arises, for example, when locating a transit line for commuters with the aim of making the line easily accessible to all the clients scattered in a given territory.

The weighted Range function can be considered as a generalization of the weighted center criterion, in the sense that it tries to overcome some problems related to the eccentricity. In fact, it is well-known, (see, [13]), that the weighted center criterion tends to favor few clients that are concentrated in a given geographical area, far from the facility, to the detriment

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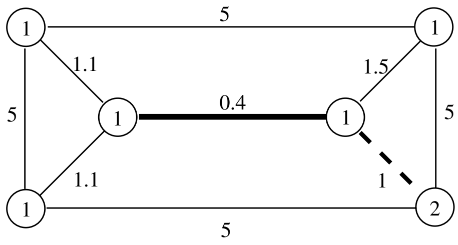


**Fig. 1.** An example of the different locations of a path when using the weighted Range criterion or the weighted eccentricity criterion. Assume to locate a path connecting two vertices and with length at most 1. The path minimizing the weighted Range function is in bold, while the one minimizing the weighted eccentricity is represented by a dashed line.

**Table 1**
Summary of results.

| Range-type problems | | Unweighted case | | Weighted case | | Length constraints | |
|---|---|---|---|---|---|---|---|
| Problem | | Discrete | Continuous | Discrete | Continuous | Discrete | Continuous |
| P1 | $\min R(P)$ | $O(n)$ | $O(n^2)$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^3)}$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^3)}$ |
| P2 | $\min\ E(P)$ s.t. $\mu(P) \geq \gamma$ | $O(n)$ | $O(n)$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^3)}$ |
| P3 | $\max\ \mu(P)$ s.t. $E(P) \leq \gamma$ | $O(n)$ | $O(n)$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^2)}$ | $\mathbf{O(n^3)}$ |

of most clients closer to the facility but dispersed in the rest of the territory. On the contrary, the weighted Range objective function takes into account also clients "close" to the facility with the final aim of guaranteeing that all clients are fairly treated. Fig. 1 provides a simple example of the location of a path in a given network, with both the weighted Range and the weighted center criterion, showing that the former provides a "more central" facility than the latter.

Some recent papers [5,19] introduced other objectives and criteria, such as the *variance* criterion, related to the variability of the distribution of the distances from the demand points to a facility. Actually, equity problems have been already considered in the literature when facing the classical point location problems. In particular, the minimum Range single point location has been introduced in [15]. The extension to the case of locating path-shaped facilities by using the Range criterion was provided in [18] where the problem was formulated in the following three different versions: locating a path which minimizes the Range, locating a path which minimizes the maximum distance subject to the minimum distance bounded below by a constant, and locating a path which maximizes the minimum distance subject to the maximum distance bounded above by a constant. In [18], it is shown that these problems are NP-Hard for general networks, while for tree networks efficient polynomial time algorithms are provided. However, in [18] all these problems were studied in the special case in which all the vertices of the network have the same weight. Here we extend our previous results to the more general case of arbitrary vertex weights. It is well-known, (see, [25]), that the introduction of arbitrary weights for the vertices makes location problems much difficult to solve, and completely different approaches may be required in order to get polynomial time algorithms with low complexity. Indeed, assigning non negative weights to the vertices of a network may cause the lost of some desirable properties for the distance function, like, for example, the triangular inequality.

Relying on a weighted distance function, in this paper we study the above listed Range-type problems on tree networks. We consider the discrete version of the problems, that is, when the endpoints of the located path must correspond to some vertices of the tree, and the continuous one, that is, when the endpoints of an optimal path may lie also in the interior of an edge. Moreover, all the problems are discussed by including an additional constraint on the length of the path. For the solution of these problems we provide polynomial time algorithms with low complexity (see Table 1).

The paper is organized as follows. In Section 2, we introduce the problem and some basic notation and definitions. Section 3 describes the algorithm for solving the minimum weighted Range problem on a tree both in the continuous and in the discrete versions. It is also shown that the additional constraint on the length of the path does not increase the complexity of the algorithm. In Section 4 we focus on the other two formulations of the Range problem. Although, in principle, one can obtain an algorithm for these problems with a complexity cubic in the number of the vertices of the tree by using the same algorithmic strategy of the previous section (with or without the length constraint), we show that, in the special case without the length constraint, this complexity can be lowered to be quadratic.

## 2. Notation and definitions

Let $T = (V, E)$ be a tree with $|V| = n$. Suppose that a non negative weight $w_v$ is associated to each vertex $v \in V$, while a positive real length $\ell(e) = \ell(u, v)$ is assigned to each edge $e = (u, v) \in E$. When $T$ is rooted at a vertex $r$ it is denoted by $T_r$. We denote by $V(T_r)$ the set of vertices of $T_r$. For any vertex $v$, let $T_v$ be the subtree of $T_r$ rooted at vertex $v$, $S(v)$ the set of children of $v$ in $T_r$, and $p(v)$ the parent of $v$ in $T_r$. Clearly, a vertex $v$ is a leaf if and only if $|S(v)| = 0$. Given an edge $(u, v)$ in $T_r$, suppose that $v = p(u)$, then for a point $x$ in the interior of $(u, v)$, we also refer to $x$ as the distance $d(u, x)$. For any pair of points $x$ and $y$ in $T$, that may be vertices or may belong to the interior of an edge, we denote by $P(x, y)$ the unique path connecting $x$ and $y$. We denote by $L(P(x, y))$ the length of $P(x, y)$. In the following, we will avoid to specify one or both the endpoints of a path when it is not necessary. A path $P$ is *discrete* if both its endpoints are vertices of $T$, otherwise it is *continuous*. We denote by $V(P)$ the set of vertices belonging to $P$. Let $d(u, P)$ be the distance from a vertex $u$ to a path $P$, that is, the length of the shortest path from $u$ to a vertex or an endpoint of $P$. For any point $x$ in $T$, the weighted eccentricity of $x$ is $E(x) = \max_{u\in V} w_u d(u, x)$, while for any path $P$ the weighted eccentricity of $P$ is $E(P) = \max_{u\in V} w_u d(u, P)$. We also denote by $\mu(x)$ and $\mu(P)$ the minimum weighted distance from a vertex $u$ to $x$ and the minimum weighted distance from a vertex $u$ to $P$, respectively. One has $\mu(x) = \min_{u\in V\setminus x} w_u d(u, x)$ if $x$ is a vertex, and $\mu(x) = \min_{u\in V} w_u d(u, x)$ if $x$ is a point, while for a path $P$ one has $\mu(P) = \min_{u\in V\setminus V(P)} w_u d(u, P)$.

For a tree $T = (V, E)$, we consider the weighted Range objective function which is a non negative variability measure defined as follows:

$$R(P) = \max_{u\in V\setminus V(P)} w_u d(u, P) - \min_{u\in V\setminus V(P)} w_u d(u, P). \tag{1}$$

Given a path $P$, since $d(u, P) = 0$ for each $u \in V(P)$, (1) can be rewritten in the equivalent form:

$$R(P) = E(P) - \mu(P). \tag{2}$$

In this paper we investigate the problem of finding a path $P$ in $T$ that minimizes the weighted Range function (problem P1). We will study both the discrete and the continuous cases also considering the constrained version of the problems with a bound on the length of the path. For the unconstrained version, we suppose that the tree $T$ is not a path, since otherwise one can always assume that the optimal path is the tree itself (see, [18]). On the other hand, when there is a bound on the length of the path, the problem is not trivial even if $T$ is a path.

As in [18], we also study the following two related problems, that we call P2 and P3, respectively:

$$\begin{aligned} &\min\ E(P) \\ &\text{s.t. } \mu(P) \geq \gamma \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\max\ \mu(P) \\ &\text{s.t. } E(P) \leq \gamma \end{aligned} \tag{4}$$

where $\gamma$ is a given non negative constant.

These problems were already formulated and solved in [18] for the location of a path in the particular case when all the vertices of the tree $T$ have the same weight, providing linear time algorithms for all of them but one which was instead proved to be solvable in quadratic time (see Table 1). Here we develop the analysis for the more general case of arbitrary non negative weights associated to the vertices of $T$. As we will see, the introduction of vertex weights makes the problem much difficult to solve and a completely different approach is required in order to get polynomial time algorithms with low complexity. Table 1 reports a summary of the complexity results related to the algorithms proposed in [18] and the new results provided in the present paper (in bold), showing that in the more general case the algorithm complexity increases by one order of magnitude. Moreover, the continuous version of the unconstrained problem P1 shows to be more difficult than the others, independently from the vertex weights or the length constraint. In this paper we also study the Range minimization problems P1, P2 and P3 with an additional constraint on the length of the path for which we show that the time complexity is cubic in the most difficult case related to the location of continuous paths. However, for the discrete version of these problems, we are still able to provide $O(n^2)$ solution algorithms.

## 3. The general solution approach for the minimization of the Range objective function

Consider the problem of finding in $T$ a continuous path that minimizes the weighted Range objective function (i.e., problem P1). The idea of the algorithm is to root the tree at a vertex $r$, and consider the rooted tree $T_r$. Then, for every pair of edges $(u, v)$ and $(k, h)$ in $T_r$, we find the path $P(x, y)$ with $x \in (u, v)$ and $y \in (k, h)$ that minimizes the Range function. The optimal solution for problem P1 is a path $P^*$ that minimizes $R(P)$ among all the paths $P(x, y)$, with $x, y$ varying in two edges of $T$. In order to find $P^*$, for every edge $(u, v)$, with $v = p(u)$, the algorithm evaluates all the possible edges $(k, h)$ by visiting the two subtrees $T_u$ and $T_{p(u)}$ that can be obtained from $T_r$ by removing $(u, p(u))$. Note that $T_{p(u)}$ is the subtree of $T$ whose set of vertices is $V \setminus V(T_u)$ (see Fig. 2).

### 3.1. Basic formulas

Consider a vertex $u$ in $T_r$, and let $S(u) = \{s_1, s_2, \ldots, s_{m_u}\}$ be the set of the children of $u$, with $|S(u)| = m_u$. First, we compute the maximum and the minimum weighted distances from vertex $u$ to the vertices in each subtree $T_{s_i}$, $E^{s_i}(u)$ and

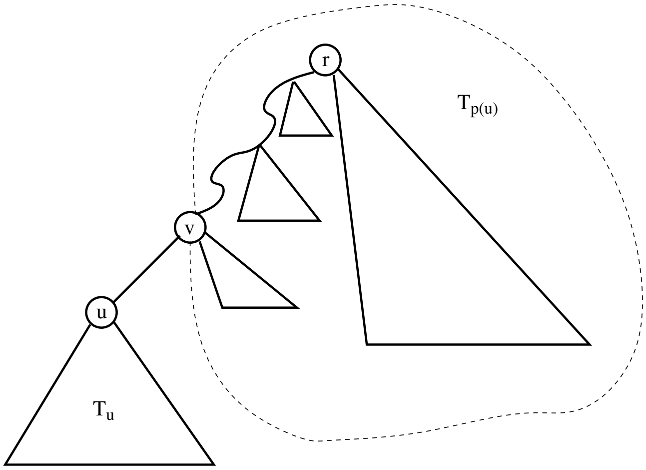


**Fig. 2.** The subtree $T_u$ and the subtree $T_{p(u)}$ that can be obtained from $T_r$ by removing $(u, v)$.

$\mu^{s_i}(u),\ i = 1, \ldots, m_u$ as follows:

$$E^{s_i}(u) = \max_{z \in V(T_{s_i})} w_z d(u, z),$$

$$\mu^{s_i}(u) = \min_{t \in V(T_{s_i})} w_t d(u, t).$$

Furthermore we need to compute $E^{p(u)}(u)$ and $\mu^{p(u)}(u)$ that denote the maximum and minimum weighted distances from $u$ to the vertices in $T_{p(u)}$:

$$E^{p(u)}(u) = \max_{z \in V(T_{p(u)})} w_z d(u, z),$$

$$\mu^{p(u)}(u) = \min_{t \in V(T_{p(u)})} w_t d(u, t).$$

The following result is straightforward.

**Proposition 1.** *For all the vertices $u$ in $T_r$, the quantities $E^{s_i}(u)$, $\mu^{s_i}(u)$, $i = 1, \ldots, m_u$, and $E^{p(u)}(u)$, $\mu^{p(u)}(u)$ can be computed once in $O(n^2)$ time.*

On the basis of the above quantities, for any given $s \in S(u)$, we define the eccentricity of $u$ w.r.t. the vertices in $V(T_u) \backslash V(T_s)$ as follows:

$$E_s(u) = \begin{cases} \max\limits_{\substack{s_i \in S(u) \\ s_i \neq s}} E^{s_i}(u) & \text{if } |S(u)| \geq 2 \\ 0 & \text{if } |S(u)| = 0 \text{ or } 1. \end{cases} \tag{5}$$

Similarly, for the minimum distance from $u$ w.r.t. the vertices in $V(T_u) \setminus V(T_s)$, one has:

$$\mu_s(u) = \begin{cases} \min\limits_{\substack{s_i \in S(u) \\ s_i \neq s}} \mu^{s_i}(u) & \text{if } |S(u)| \geq 2 \\ +\infty & \text{if } |S(u)| = 0 \text{ or } 1. \end{cases} \tag{6}$$

These quantities will be exploited in the main procedure to solve the weighted Range problems. Actually, for a given $u$ in $T_r$ such that $|S(u)| \geq 2$, we compute:

$$E_0(u) = \max_{s_i \in S(u)} E^{s_i}(u); \qquad s_{\max} \in \operatorname{argmax}\{E^{s_i}(u) : s_i \in S(u)\}, \tag{7}$$

and

$$E_{s_{\max}}(u) = \max_{\substack{s_i \in S(u) \\ s_i \neq s_{\max}}} E^{s_i}(u). \tag{8}$$

When $|S(u)| = 1$, one has $E_0(u) = E^{s_1}(u)$ and $E_{s_{\max}}(u) = 0$, while, when $|S(u)| = 0$, we set $E_0(u) = E_{s_{\max}}(u) = 0$. Notice that $E_0(u)$ and $E_{s_{\max}}(u)$ are the maximum and the second maximum weighted distances from $u$ to the vertices in $\bigcup_{i=1,\ldots,m_u} V(T_{s_i})$, respectively. Similarly, for the minimum distance we compute:

$$\mu_0(u) = \min_{s_i \in S(u)} \mu^{s_i}(u); \qquad s_{\min} \in \operatorname{argmin}\{\mu^{s_i}(u) : s_i \in S(u)\}, \tag{9}$$

and

$$\mu_{s_{\min}}(u)=\min_{\substack{s_i\in S(u)\\ s_i\neq s_{\min}}}\mu^{s_i}(u). \tag{10}$$

When $|S(u)|=1$, one has $\mu_0(u)=\mu^{s_1}(u)$ and $\mu_{s_{\min}}(u)=+\infty$, while, when $|S(u)|=0$, we set $\mu_0(u)=\mu_{s_{\min}}(u)=+\infty$. Quantities $\mu_0(u)$ and $\mu_{s_{\min}}(u)$ are the minimum and the second minimum weighted distances from $u$ to the vertices in $\bigcup_{i=1,\dots,m_u} V(T_{s_i})$, respectively.

Then, if $|S(u)|\geq 2$, the computation of (5) and (6) relies only on quantities (7)–(10), and one has:

$$E_s(u)=\begin{cases}E_0(u) & \text{if } s\neq s_{\max}\\ E_{s_{\max}}(u) & \text{if } s=s_{\max}\end{cases} \tag{11}$$

and

$$\mu_s(u)=\begin{cases}\mu_0(u) & \text{if } s\neq s_{\min}\\ \mu_{s_{\min}}(u) & \text{if } s=s_{\min}.\end{cases} \tag{12}$$

**Proposition 2.** *For all the vertices $u$ in $T_r$ and all the children of $u, s_i,\ i=1,\dots,m_u$, the quantities $E_{s_i}(u)$ and $\mu_{s_i}(u)$, can be computed once in $O(n^2)$ time.*

### 3.2. The algorithm

Consider two fixed edges $(u,v)$ and $(k,h)$ in $T_r$. In this section we explain how the algorithm finds a continuous path $P(x,y)$, with $x\in(u,v)$ and $y\in(k,h)$ (possibly $(u,v)=(k,h)$) that minimizes the weighted Range objective function. The general idea of the algorithm is to fix one edge, say $(u,v)$, and to evaluate all the possible paths $P(x,y)$ with $x\in(u,v)$ and $y$ in any other edge $(k,h)$ in $T_r$. Three different cases are possible:

(i) $(k,h)$ is in $T_u$;
(ii) $(k,h)$ is in $T_{p(u)}$ and belongs to the path from $v=p(u)$ to the root $r$;
(iii) $(k,h)$ belongs to a subtree of $T_{p(u)}$ rooted at a vertex belonging to the path from $v=p(u)$ to the root $r$.

The procedure is repeated for all the possible edges $(u,v)$ in $T_r$.

Consider a fixed edge $(u,v)$. We first analyze case (i) related to a continuous path $P(x,y)$ with $x$ belonging to $(u,v)$ and $y$ belonging to an edge $(k,h)$ in $T_u$ (see Fig. 3). In the rest of this section, for a given $(k,h)$, we assume that $h=p(k)$ and for a point $y$ in $(k,h)$, we refer to $y$ also as the distance $d(k,y)$. In case (i), we evaluate all edges $(k,h)$ in $T_u$ by a top-down, level-by-level, visit of $T_u$. In order to compute the weighted Range of $P(x,y)$, say $R(P(x,y))$, we need the maximum and the minimum weighted distances $E(P(x,y))$ and $\mu(P(x,y))$ in $T_r$.

The maximum weighted distance $E(P(x,y))$ can be attained at $y$, at $x$, or at some vertices along $P(x,y)$, and, therefore, it is given by the maximum among the following three quantities:

$$E_y=\max_{z\in V(T_k)}\{w_z d(z,k)+w_z y\}, \tag{13}$$

$$E_x=\max_{z\in V(T_{p(u)})}\{w_z d(z,v)+w_z[\ell(u,v)-x]\}, \tag{14}$$

$$E_{P(u,h)}=\max\{\max_{\substack{d,q\in V(P(u,h))\\ d=p(q);d\neq h}}E_q(d);\ E_k(h)\}, \tag{15}$$

that is:

$$E(P(x,y))=\max\{E_y,E_x,E_{P(u,h)}\}. \tag{16}$$

Similarly, for $\mu(P(x,y))$ we have:

$$\mu(P(x,y))=\min\{\mu_y,\mu_x,\mu_{P(u,h)}\}, \tag{17}$$

where:

$$\mu_y=\min_{t\in V(T_k)}\{w_t d(t,k)+w_t y\}, \tag{18}$$

$$\mu_x=\min_{t\in V(T_{p(u)})}\{w_t d(t,v)+w_t[\ell(u,v)-x]\}, \tag{19}$$

$$\mu_{P(u,h)}=\min\{\min_{\substack{d,q\in V(P(u,h))\\ d=p(q);d\neq h}}\mu_q(d);\ \mu_k(h)\}. \tag{20}$$

For the two edges $(u,v)$ and $(k,h)\in T_u$, we can find the two endpoints $\bar{x}\in(u,v)$ and $\bar{y}\in(k,h)$ that minimize $R(P(x,y))$ by solving a suitable linear program. We distinguish two cases, that is, when $(u,v)=(k,h)$- i.e., both $x$ and $y$ belong to the

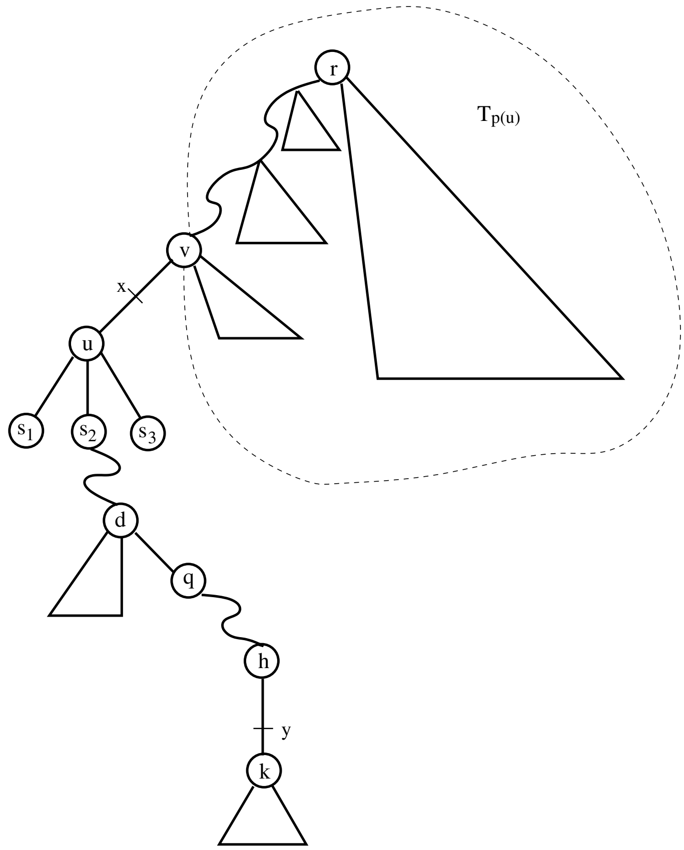


**Fig. 3.** An example of a path $P(x, y)$ with $x \in (u, v)$ and $y \in (k, h)$ in $T_u$ (case (i)).

same edge, say $(u, v)$- and when $(u, v) \neq (k, h)$. In the first case, considering formulas (13)–(14) and (18)–(19), we have to solve the following problem:

$$\begin{aligned}
&\min_{E,\mu,x,y} \; E - \mu \\
&\text{s.t.} \\
&w_z d(z, k) + w_z y \le E \quad \forall z \in V(T_k) \\
&w_z d(z, v) + w_z\big[\ell(u, v) - x\big] \le E \quad \forall z \in V(T_{p(u)}) \\
&w_t d(t, k) + w_t y \ge \mu \quad \forall t \in V(T_k) \\
&w_t d(t, v) + w_t\big[\ell(u, v) - x\big] \ge \mu \quad \forall t \in V(T_{p(u)}) \\
&y \le x \\
&E, \mu, x, y \ge 0.
\end{aligned} \tag{21}$$

When $(u, v) \neq (k, h)$, it suffices to add to problem (21) the following two constraints:

$$\begin{aligned}
&E_{P(u,h)} \le E \\
&\mu_{P(u,h)} \ge \mu
\end{aligned} \tag{22}$$

where $E_{P(u,h)}$ and $\mu_{P(u,h)}$ are defined by formulas (15) and (20), respectively.

In both versions, the linear program has four variables and $O(n)$ constraints, so that it can be solved in $O(n)$ time [14]. In order to formulate problem (21) with constraints (22) the two bounds $E_{P(u,h)}$ and $\mu_{P(u,h)}$ must be available. We want to emphasize here that, for each pair of edges $(u, v)$ and $(k, h)$ in $T_u$, computing these two quantities from scratch would require $O(n^2)$ time. However, exploiting formulas (11) and (12), the same can be done in constant time during the top-down visit of $T_u$. Let $(k, h)$ be the current edge and let $(h, a)$ be the edge in $T_u$ with $a = p(h)$. When moving top-down from the edge $(h, a)$ to the edge $(k, h)$, the quantities $E_{P(u,h)}$ and $\mu_{P(u,h)}$ can be updated from the corresponding $E_{P(u,a)}$ and $\mu_{P(u,a)}$ as follows:

$$E_{P(u,h)} = \max\{E_{P(u,a)}, \max_{\substack{s_i \in S(h) \\ s_i \neq k}} E^{s_i}(h)\} = \max\{E_{P(u,a)}, E_k(h)\} \tag{23}$$

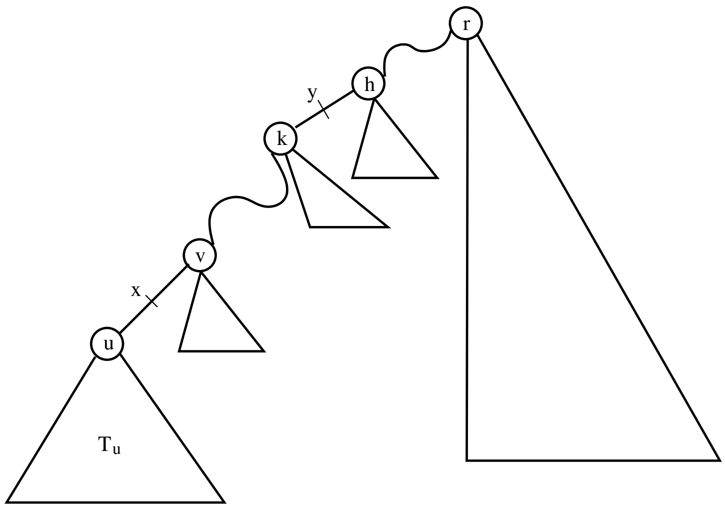


**Fig. 4.** An example of a path $P(x, y)$ with $y \in (k, h)$ belonging to the path $P(v, r)$ (case (ii)).

and

$$\mu_{P(u,h)} = \min\{\mu_{P(u,a)}, \min_{\substack{s_i \in S(h) \\ s_i \neq k}} \mu^{s_i}(h)\} = \min\{\mu_{P(u,a)}, \mu_k(h)\}. \tag{24}$$

When $u = h$, the subpath $P(u, h)$ corresponds to the single vertex $u$ and the above formulas are initialized as $E_{P(u,h)} = E_k(u) = E_k(h)$ and $\mu_{P(u,h)} = \mu_k(u) = \mu_k(h)$, respectively.

We now analyze case (ii) of a continuous path $P(x, y)$ with $x \in (u, v)$ and $y \in (k, h)$, with $h = p(k)$, and $(k, h)$ belonging to the path $P(v, r)$ from $v$ to the root $r$ (see, Fig. 4). In this case, we evaluate all the candidate edges $(k, h)$ by a bottom-up visit of $P(v, r)$.

As before, given a path $P(x, y)$, we have to evaluate $E(P(x, y))$ and $\mu(P(x, y))$ and we formulate problem (21) with constraints (22) for each possible $(k, h)$. Here we compute $E_x$, $E_y$, and $E_{P(v,k)}$ as follows. The maximum weighted distance w.r.t. $x$ is attained through vertex $u$ and it is given by:

$$E_x = \max_{z \in V(T_u)} \{w_z d(z, u) + w_z x\}. \tag{25}$$

For the maximum weighted distance w.r.t. the subpath $P(v, k)$, we have:

$$E_{P(v,k)} = \max_{\substack{d,q \in V(P(u,k)) \\ d=p(q); d \neq u}} E_q(d). \tag{26}$$

Moreover, we have:

$$E_y = \max_{z \in V(T_{p(k)})} \{w_z d(z, h) + w_z[\ell(k, h) - y]\}. \tag{27}$$

Similarly, we can compute all the quantities for determining $\mu_{P(x,y)}$ as follows:

$$\mu_x = \min_{t \in V(T_u)} \{w_t d(t, u) + w_t x\}. \tag{28}$$

$$\mu_{P(v,k)} = \min_{\substack{d,q \in V(P(u,k)) \\ d=p(q); d \neq u}} \mu_q(d), \tag{29}$$

and

$$\mu_y = \min_{t \in V(T_{p(k)})} \{w_t d(t, h) + w_t[\ell(k, h) - y]\}. \tag{30}$$

Visiting the path from $v$ to $r$ in $T_r$ bottom-up, let $(k, h)$ be the current edge and let $(b, k)$ be the edge in $P(v, r)$ with $k = p(b)$. When moving bottom-up from the edge $(b, k)$ to the new edge $(k, h)$, the quantities $E_{P(v,k)}$ and $\mu_{P(v,k)}$ can be updated from the corresponding $E_{P(v,b)}$ and $\mu_{P(v,b)}$ in constant time using the following recursive formulas:

$$E_{P(v,k)} = \max\{E_{P(v,b)}, E_b(k)\} \tag{31}$$

and

$$\mu_{P(v,k)} = \min\{\mu_{P(v,b)}, \mu_b(k)\}. \tag{32}$$

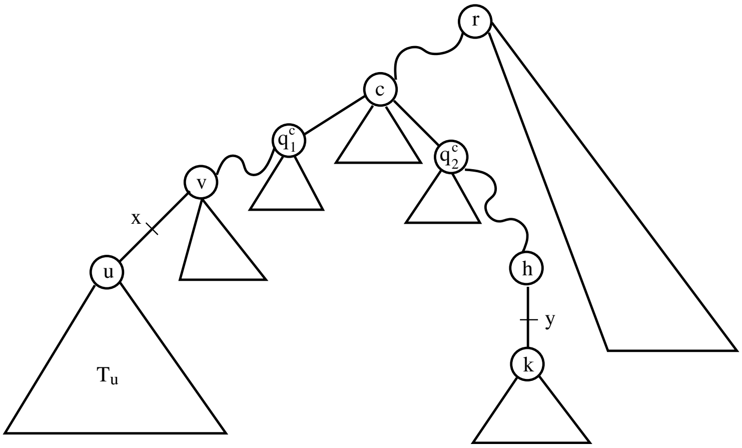


**Fig. 5.** An example of a path $P(x, y)$ with $y \in (k, h)$ belonging to the subtree of $T_{p(u)}$ rooted at a vertex $c$ in $P(v, r)$ (case (iii)).

When $v = k$, the path $P(v, k)$ corresponds to the single vertex $v$ and the above formulas are initialized at $E_{P(v,k)} = E_u(k) = E_u(v)$ and $\mu_{P(v,k)} = \mu_u(k) = \mu_u(v)$, respectively.

In order to find the path $P(\bar{x}, \bar{y})$ that minimizes $R(P(x, y))$ among all the paths $P(x, y)$ with $x$ belonging to $(u, v)$, and $y$ belonging to $(k, h)$ lying in the path from $v$ to $r$, one can still formulate and solve a linear program analogous to (21) with the additional constraints (22) in $O(n)$ time.

Consider now case (iii) where $y \in (k, h)$, with $h = p(k)$ and $(k, h)$ belonging to a subtree of $T_{p(u)}$ rooted at a vertex in the path from $v = p(u)$ to the root $r$ (see, Fig. 5).

As before, it is necessary to evaluate $E_x$, $E_y$ and $E_{P(v,h)}$ appropriately. First of all, notice that formulas (25) and (28) can be still used for computing $E_x$ and $\mu_x$. Also for the quantities $E_y$ and $\mu_y$, we can refer directly to formulas (13) and (18). It remains to consider the maximum and minimum weighted distances w.r.t. the subpath $P(v, h)$, i.e., the quantities $E_{P(v,h)}$ and $\mu_{P(v,h)}$, respectively. Let $c$ be the least common ancestor in $T_r$ of the vertices $u$ and $k$. Notice that, the least common ancestor of every pair of vertices in a given rooted tree with $n$ vertices can be computed once in $O(n)$ [3]. Let $(q_1^c, c)$ be the last edge in the path $P(v, c) \subseteq P(v, h)$ and $(q_2^c, c)$ be the first edge along $P(c, h) \subseteq P(v, h)$ (Fig. 5). Notice that $c$ may coincide with the root $r$. Then, we have:

$$E_{P(v,h)} = \max\{E_{P(v,q_1^c)},\ \max_{\substack{s_i\in S(c)\\ s_i\neq q_1^c,q_2^c}} E^{s_i}(c),\quad E^{p(c)}(c),\quad E_{P(q_2^c,h)}\}, \tag{33}$$

and:

$$\mu_{P(v,h)} = \min\{\mu_{P(v,q_1^c)},\ \min_{\substack{s_i\in S(c)\\ s_i\neq q_1^c,q_2^c}} \mu^{s_i}(c),\quad \mu^{p(c)}(c),\quad \mu_{P(q_2^c,h)}\}. \tag{34}$$

Notice that, in case (iii), for a fixed $(u, v)$ and a vertex $c$ belonging to $P(v, r)$, the candidate edges $(k, h)$ are evaluated by a top-down visit of the tree induced by the vertices in $V(T_c) \setminus V(T_{q_1^c})$. During this visit, in formulas (33) and (34) the quantities $E_{P(v,q_1^c)}$ and $\mu_{P(v,q_1^c)}$ are recursively updated in constant time by formulas similar to (31) and (32), respectively. For $E_{P(q_2^c,h)}$ and $\mu_{P(q_2^c,h)}$ formulas analogous to (23) and (24) can be adopted. Moreover, notice that, if at the beginning one computes in $T_r$ also the *third* maximum and minimum weighted distances from any vertex $u \in V$ to the vertices in $\bigcup_{i=1,\dots,m_u} V(T_{s_i})$, then, for $c$ such that $|S(c)| \geq 3$, the quantities

$$\max_{\substack{s_i\in S(c)\\ s_i\neq q_1^c,q_2^c}} E^{s_i}(c), \quad \text{and} \quad \min_{\substack{s_i\in S(c)\\ s_i\neq q_1^c,q_2^c}} \mu^{s_i}(c)$$

are automatically available when formulas (33) and (34) must be computed. When $|S(c)| < 3$, the above two quantities must not be computed at all.

In the special case when $c = r$ the two quantities $E^{p(c)}(c)$ and $\mu^{p(c)}(c)$ do not appear in (33) and (34).

The above discussion shows that the complexity of the algorithm does not increase in cases (ii) and (iii) w.r.t. case (i). Therefore, once all the necessary quantities have been computed, the following complexity result holds.

**Proposition 3.** *The time complexity for computing the weighted Range of a path $P(x, y)$ with $x$ belonging to a fixed edge $(u, v)$ and $y$ to another fixed edge $(k, h)$ in $T_r$ is $O(n)$. Hence, the overall time complexity for computing the path $P$ in $T$ that minimizes the Range function is $O(n^3)$.*

The previous approach can also be used for solving the more general problem of finding a continuous path $P$ that minimizes the Range objective function with an additional constraint on the length of $P$, which we denote by $L(P) =$

$\sum_{e\in P}\ell(e)$, that is, $L(P)\leq\hat{L}$. Consider for example the case when $(k,h)\in T_u$ (the other cases are similar), the approach is exactly the same as before but, when solving (21) with constraints (22) one must first check if $L(P(u,h))$ is strictly less than $\hat{L}$, since otherwise we do not need to solve the problem because we already know that it is infeasible. In the particular case when $L(P(u,h))=\hat{L}$, the only feasible path is $P(x,y)=P(u,h)$, while if $L(P(u,h))<\hat{L}$, it suffices to consider also the following additional constraint:

$$L(P(u,h))+x+\ell(k,h)-y\leq\hat{L},$$

that is,

$$x-y\leq\hat{L}-L(P(u,h))-\ell(k,h).$$

Since the discrete weighted path Range problem is a special case of the continuous one, the above approach can be still applied to the former problem with an overall time complexity of $O(n^2)$ due to the fact that, in the discrete case, we do not need to formulate and solve any linear program. Indeed, for any two vertices $u$ and $h$ in $T_r$, whichever the configuration of the discrete path $P(u,h)$ (i.e., cases (i),(ii), or (iii)), $E(P(u,h))$ and $\mu(P(u,h))$ can be computed by using recursive formulas similar to those introduced at the beginning of this section. In particular, formulas discussed in Section 3.1 can be applied in a preprocessing phase to compute, for all $u$ in $T_r$, $E^{s_i}(u)$ and $\mu^{s_i}(u)$, $i=1,\ldots,m_u$; $E^{p(u)}(u)$, $\mu^{p(u)}(u)$; $E_{s_i}(u)$ and $\mu_{s_i}(u)$ for all children of $u$. By Propositions 1 and 2 these quantities can be obtained once in $O(n^2)$ time. In addition, as already discussed above, $E_{P(u,h)}$ and $\mu_{P(u,h)}$ can be updated in constant time by using formulas (23) and (24), respectively. Thus, given a vertex $u\in V(T_r)$, finding the weighted Range of a path $P(u,h)$ for all $h\in V(T_r)$ requires $O(n)$ time, and, therefore, the overall time complexity for solving the discrete version of the Range problem P1 is $O(n^2)$. The additional constraint on the length of the optimal path does not increase such complexity.

To conclude, we note that the algorithm described in this section could be applied also to solve the continuous version of the two constrained Range-type problems P2 and P3, with or without the length constraint, in time $O(n^3)$. However, in the following section, we show that, when there is no length constraint, also the continuous versions of P2 and P3 can be solved more efficiently in time $O(n^2)$.

## 4. Problems P2 and P3

In this section, we discuss both the discrete and the continuous versions of the constrained problems P2 and P3, and show that both can be solved in $O(n^2)$ time. In the discrete case, both P2 and P3 can be solved by using the same algorithm described at the end of Section 3.2, with the additional task of performing a check for feasibility on each evaluated path with respect to $\mu(P)$ in P2 and to $E(P)$ in P3. For the continuous versions of the two problems, in the following we will show that they can be reformulated as equivalent discrete problems defined on a new augmented tree $\bar{T}$ with $O(n)$ additional vertices w.r.t. $T$. This implies that also the two continuous problems P2 and P3 can be solved with the same $O(n^2)$ complexity by applying to $\bar{T}$ the discrete-case algorithm.

Consider the continuous version of problem P2, that, for a given $\gamma\geq 0$ consists of finding the two points $x$ and $y$ such that:

$$\min_{x,y}\ E(P(x,y))$$
$$\text{s.t. } \mu(P(x,y))\geq\gamma.$$

For a point $y$ in $(k,h)$, here we refer to $y$ also as the distance from the point $y$ to vertex $k$ (see Fig. 6). We also consider $(k,h)$ as a continuous interval of values. For a given edge $(k,h)$ in $T$ we denote by $T(k)$ and $T(h)$ the two subtrees that can be obtained by removing $(k,h)$ from $T$, and containing $k$ and $h$, respectively (see, Fig. 6). As before, the set of vertices of $T(k)$ and $T(h)$ are denoted by $V(T(k))$ and $V(T(h))$, respectively.

**Proposition 4.** *Let $P(\bar{x},h)$ be a path in $T$ with $\bar{x}$ fixed in $T(h)$, and such that $\mu(P(\bar{x},h))\geq\gamma$ and $\mu(P(\bar{x},k))<\gamma$. Then, there exists a unique point $\bar{y}$ in $(k,h)$ such that:* (a) *$\mu(P(\bar{x},\bar{y}))=\gamma$;* (b) *for every $y\neq\bar{y}$ in $(k,h)$ satisfying $\mu(P(\bar{x},y))\geq\gamma$, one has:*

$$\mu(P(\bar{x},y))\geq\mu(P(\bar{x},\bar{y})),\quad \textit{and}\quad E(P(\bar{x},y))\geq E(P(\bar{x},\bar{y})).$$

**Proof.** For every $z$ in $T(k)$ the weighted distance from $z$ to $P(\bar{x},y)$, with $y$ in $(k,h)$, is given by:

$$w_z d(z,y)=w_z d(z,k)+w_z y,$$

and it is a continuous monotone non decreasing function w.r.t. $y$ in $(k,h)$. This implies that $E(P(\bar{x},y))$ and $\mu(P(\bar{x},y))$ are continuous monotone non decreasing functions in $y$ as well. Hence, due to continuity, there must exist a point $\bar{y}$ in $(k,h)$ such that $\mu(P(\bar{x},\bar{y}))=\gamma$, and $\forall y\neq\bar{y}$ in $(k,h)$ for which $\mu(P(\bar{x},y))\geq\gamma$, one has $E(P(\bar{x},y))\geq E(P(\bar{x},\bar{y}))$.

Notice that, when $\mu(P(\bar{x},k))\geq\gamma$, one always has $\bar{y}=k$. □

The above result is based on the fact that, once the edge $(k,h)$ and the point $\bar{x}$ are fixed (e.g. $\bar{x}\in V(T(h))$) such that $P(\bar{x},h)$ is feasible for P2, the location of $\bar{y}$ in $(k,h)$ depends only on the edge $(k,h)$ and on the subtree $T(k)$. In addition, also notice

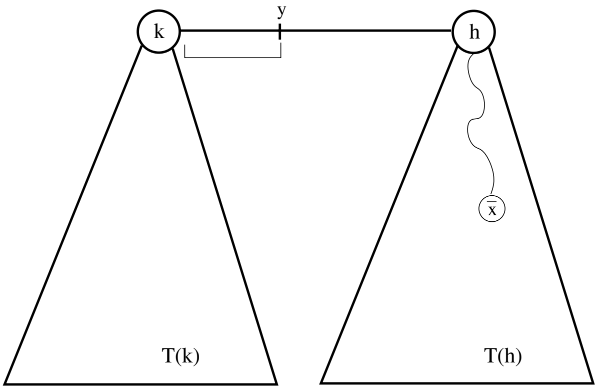


**Fig. 6.** The two subtrees $T(k)$ and $T(h)$ obtained by deleting the edge $(k, h)$, and a point at distance $y$ from vertex $k$.

that the location of $\bar{y}$ in $(k, h)$ does not depend on $\bar{x}$, but only on the fact that $\bar{x}$ is in $T(h)$, that is, the same location of $\bar{y}$ in $(k, h)$ holds for all $P(\bar{x}, \bar{y})$, whichever the location of $\bar{x}$ in $T(h)$.

Obviously, the same result applies if one fixes $\bar{x}$ arbitrarily in $T(k)$ and searches for a path $P(\bar{x}, \bar{y})$ with $\bar{y}$ in $(k, h)$ that minimizes the eccentricity function under the assumption that $\mu(P(\bar{x}, k)) \geq \gamma$. Also in this case, when $\mu(P(\bar{x}, h)) \geq \gamma$, one always has $\bar{y} = h$.

The above discussion implies that, besides $k$ and $h$, there are only two points in $(k, h)$ that are "relevant" for P2 and one can always augment the set of vertices of $T$ by adding them to $V$. We call such new vertices *semivertices*, and we denote by $\bar{T}$ the corresponding augmented tree. The optimal solution of the continuous version of problem P2 in $T$ corresponds to the optimal solution of the discrete version of the same problem P2 in $\bar{T}$. Thus, the continuous optimal solution can be found by applying to $\bar{T}$ the discrete version of the algorithm described in Section 3.2, provided that at each step a check for feasibility is performed on the current path $P$.

In the following we explain how the two semivertices can be located in any given edge of $T$. Consider edge $(k, h)$ and suppose that $\bar{x} \in V(T(h))$. In order to find the semivertex $\bar{y} \in (k, h)$, one must solve the following linear program:

$$\begin{aligned} & \min\ y \\ & \text{s.t.} \\ & w_z d(z, k) + w_z y \geq \gamma \quad \forall z \in V(T(k)) \\ & 0 \leq y \leq \ell(k, h). \end{aligned} \tag{35}$$

Problem (35) is a linear program with one variable and $O(n_k)$ constraints, with $n_k = |V(T(k))|$, and it can be solved in time $O(n_k)$ [14]. A similar problem can be formulated to find the semivertex in $(k, h)$ related to $\bar{x} \in V(T(k))$ in time $O(n_h)$, $n_h = |V(T(h))|$. It follows that the overall time complexity for augmenting the tree $T$ is $O(n^2)$.

We now analyze the special case of a path $P(x, y)$ with both $x$ and $y$ in $(k, h)$. In this case, we can find the path $P(\bar{x}, \bar{y})$ by directly solving the following linear program.

$$\begin{aligned} & \min\ E \\ & \text{s.t.} \\ & E \geq w_z d(z, k) + w_z y \geq \gamma \quad \forall z \in V(T(k)) \\ & E \geq w_t d(t, h) + w_t \big[\ell(k, h) - x\big] \geq \gamma \quad \forall t \in V(T(h)) \\ & 0 \leq y \leq x \leq \ell(k, h). \end{aligned} \tag{36}$$

Problem (36) is a linear program with three variables and $O(n)$ constraints and it can be solved in linear time for each edge in $T$. The above discussion leads to the following result.

**Proposition 5.** *The overall time complexity for solving the continuous version of problem* P2 *is* $O(n^2)$.

Consider now the continuous version of problem P3, that, for a given $\gamma \geq 0$, consists of finding the two points $x$ and $y$ such that:

$$\begin{aligned} & \max_{x,y}\ \mu(P(x, y)) \\ & \text{s.t. } E(P(x, y)) \leq \gamma. \end{aligned}$$

As before, consider an edge $(k, h)$ and the two subtrees that can be obtained from $T$ by removing it and containing $k$ and $h$, respectively. The following proposition holds.

**Proposition 6.** *Let $P(\bar{x}, h)$ be a path in $T$ with $\bar{x}$ fixed in $T(h)$, and such that $E(P(\bar{x}, k)) \leq \gamma$ and $E(P(\bar{x}, h)) > \gamma$. Then, there exists a unique point $\bar{y}$ in $(k, h)$ such that:* (a) *$E(P(\bar{x}, \bar{y})) = \gamma$;* (b) *for every $y \neq \bar{y}$ in $(k, h)$ satisfying $E(P(\bar{x}, y)) \leq \gamma$, one has:*

$$E(P(\bar{x}, y)) \leq E(P(\bar{x}, \bar{y})), \quad \textit{and} \quad \mu(P(\bar{x}, y)) \leq \mu(P(\bar{x}, \bar{y})).$$

**Proof.** Consider $y$ as the distance $d(y, k)$. For every $z$ in $T(k)$ the weighted distance from $z$ to $P(\bar{x}, y)$, with $y$ in $(k, h)$, is given by:

$$w_z d(z, y) = w_z d(z, k) + w_z y,$$

and it is a continuous monotone non decreasing function w.r.t. $y$ in $(k, h)$. This implies that $E(P(\bar{x}, y))$ and $\mu(P(\bar{x}, y))$ are continuous monotone non decreasing functions in $y$ as well. Hence, due to continuity, there must exist a point $\bar{y} \in (k, h)$ such that $E(P(\bar{x}, \bar{y})) = \gamma$, and $\forall y \neq \bar{y}$ in $(k, h)$ for which $E(P(\bar{x}, y)) \leq \gamma$, one has $\mu(P(\bar{x}, y)) \leq \mu(P(\bar{x}, \bar{y}))$.

Notice that, when $E(P(\bar{x}, h)) \leq \gamma$, one always has $\bar{y} = h$. □

The same result applies if one fixes $\bar{x}$ in $T(k)$ and searches for a path $P(\bar{x}, \bar{y})$ with $\bar{y}$ in $(k, h)$ that maximizes the minimum weighted distance under the assumption that $E(P(\bar{x}, h)) \leq \gamma$. In this case, when $E(P(\bar{x}, k)) \leq \gamma$, one always has $\bar{y} = k$.

In view of the above results, one can always augment the set of vertices of $T$ by adding two semivertices in each edge of $T$, thus obtaining the corresponding augmented tree $\bar{T}$. Then, the optimal solution of the continuous version of problem P3 in $T$ again corresponds to the optimal solution of the discrete version of the same problem in $\bar{T}$. Hence, the problem can be solved by applying the discrete version of the algorithm described in Section 3.2.

In order to find the semivertex $\bar{y} \in (k, h)$ when $\bar{x} \in T(h)$, one must solve the following linear program:

$$\begin{aligned} &\max\ y \\ &\text{s.t.} \\ &w_z d(z, k) + w_z y \leq \gamma \quad \forall z \in V(T(k)) \\ &0 \leq y \leq \ell(k, h). \end{aligned} \tag{37}$$

A similar problem can be formulated to find the semivertex in $(k, h)$ related to $\bar{x} \in T(k)$. As before, the overall time complexity for augmenting the tree $T$ is $O(n^2)$.

For the special case of finding a path $P(x, y)$ with both $x$ and $y$ in $(k, h)$ one can solve in $O(n)$ time the following linear problem:

$$\begin{aligned} &\max\ \mu \\ &\text{s.t.} \\ &\mu \leq w_z d(z, k) + w_z y \leq \gamma \quad \forall z \in V(T(k)) \\ &\mu \leq w_t d(t, h) + w_t\big[\ell(k, h) - x\big] \leq \gamma \quad \forall t \in V(T(h)) \\ &0 \leq y \leq x \leq \ell(k, h) \\ &\mu \geq 0. \end{aligned} \tag{38}$$

Hence, the following result holds.

**Proposition 7.** *The overall time complexity for solving the continuous version of problem* P3 *is $O(n^2)$.*

## 5. Conclusions

In this paper we dealt with extended facility location problems with equity measures as objective function. In particular, we studied the problem of locating path-shaped facilities on a tree network, with non negative weights associated to the vertices and positive lengths associated to the edges, minimizing the weighted Range objective function. We also analyzed constrained variants of the problem, such as, minimizing the maximum weighted distance to a facility subject to a lower bound on the minimum distance, and maximizing the minimum weighted distance subject to an upper bound on the maximum one. These problems were first studied in [18] for the special case in which all the vertices of the network have the same weight. In the present paper, we addressed the more general case of arbitrary non negative vertex weights. In both cases, our aim was to provide polynomial algorithms with very low complexity for a problem which is applicable in many real-life decision making contexts. However, with the introduction of weights associated to the vertices, we had to resort to a completely different approach for the solution of the problems.

With the additional results of the present paper – which also solves the more general case with a constraint on the length of the path – we finalized the analysis of a complete class of optimization problems on trees. Further research can be now driven towards other possible generalizations of the problem. For example, one can consider the location of a path-shaped facility in new classes of graphs, such as networks with cycles, the simplest of which are perhaps given by the ''outerplanar'' graphs. Location problems of this type were already studied in [12] showing that a polynomial time algorithm can be found for the location of a path in an outerplanar graph w.r.t. the median criterion. Another possibility is to maintain the tree structure of the network and study the location of subtrees minimizing the Range function. To the best of our knowledge, the above generalizations of the problem have not been studied yet in the literature. In our opinion, both problems are worthwhile: taking into account the applications, the interest relies on the fact that more connected topologies for the network or for the facilities could better fit real-life problems; on the other hand, from a theoretical viewpoint, for these problems the existence of a polynomial complexity algorithm is not still guaranteed.

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